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Solution Sheet 1

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Solution 1 (Euler angles, RPY angles, quaternions)

1. Conversion of Rotation Matrix to Euler Angles

   i.) $\mathbf{zx'z''}$ Euler Angles
   Representation as a concatenation of rotations:

$$
\begin{aligned}
R &= R_{\mathbf{z}}(\alpha) \cdot R_{\mathbf{x'}}(\beta) \cdot R_{\mathbf{z''}}(\gamma) \\
&= \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\beta) & -\sin(\beta) \\ 0 & \sin(\beta) & \cos(\beta) \end{pmatrix} \begin{pmatrix} \cos(\gamma) & -\sin(\gamma) & 0 \\ \sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \cos(\alpha)\cos(\gamma) - \sin(\alpha)\cos(\beta)\sin(\gamma) & -\cos(\alpha)\sin(\gamma) - \sin(\alpha)\cos(\beta)\cos(\gamma) & \sin(\alpha)\sin(\beta) \\ \cos(\alpha)\cos(\beta)\sin(\gamma) + \sin(\alpha)\cos(\gamma) & \cos(\alpha)\cos(\beta)\cos(\gamma) - \sin(\alpha)\sin(\gamma) & -\cos(\alpha)\sin(\beta) \\ \sin(\beta)\sin(\gamma) & \sin(\beta)\cos(\gamma) & \cos(\beta) \end{pmatrix} \\
&= \begin{pmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{pmatrix}
\end{aligned}
$$

   Coefficient-wise comparison, solve for angles:

$$
\begin{aligned}
a_z &= \cos(\beta) \Rightarrow \beta = \arccos(a_z) \\
\frac{n_z}{o_z} &= \frac{\sin(\gamma)}{\cos(\gamma)} = \tan(\gamma) \Rightarrow \gamma = \arctan\left(\frac{n_z}{o_z}\right) \\
\frac{a_x}{a_y} &= -\frac{\sin(\alpha)}{\cos(\alpha)} = -\tan(\alpha) \Rightarrow \alpha = \arctan\left(-\frac{a_x}{a_y}\right)
\end{aligned}
$$

   Be aware: We implicitly assumed $a_y \neq 0$ and $o_z \neq 0$. Matrices, that do not meet this assumption, need to be considered separately.

ii.) RPY Angles

Representation as a concatenation of rotations:

$$
\begin{aligned}
R &= R_{\mathbf{z}}(\gamma) \cdot R_{\mathbf{y}}(\beta) \cdot R_{\mathbf{x}}(\alpha) \\
&= \begin{pmatrix} \cos(\gamma) & -\sin(\gamma) & 0 \\ \sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\beta) & 0 & \sin(\beta) \\ 0 & 1 & 0 \\ -\sin(\beta) & 0 & \cos(\beta) \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{pmatrix} \\
&= \begin{pmatrix} \cos(\beta)\cos(\gamma) & \sin(\alpha)\sin(\beta)\cos(\gamma) - \cos(\alpha)\sin(\gamma) & \sin(\alpha)\sin(\gamma) + \cos(\alpha)\sin(\beta)\cos(\gamma) \\ \cos(\beta)\sin(\gamma) & \sin(\alpha)\sin(\beta)\sin(\gamma) + \cos(\alpha)\cos(\gamma) & \cos(\alpha)\sin(\beta)\sin(\gamma) - \sin(\alpha)\cos(\gamma) \\ -\sin(\beta) & \sin(\alpha)\cos(\beta) & \cos(\alpha)\cos(\beta) \end{pmatrix} \\
&= \begin{pmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{pmatrix}
\end{aligned}
$$

Coefficient-wise comparison, solve for angles:

$$
\begin{aligned}
n_z &= -\sin(\beta) \Rightarrow \beta = \arcsin(-n_z) \\
\frac{o_z}{a_z} &= \frac{\sin(\alpha)}{\cos(\alpha)} = \tan(\alpha) \Rightarrow \alpha = \arctan\left(\frac{o_z}{a_z}\right) \\
\frac{n_y}{n_x} &= \frac{\sin(\gamma)}{\cos(\gamma)} = \tan(\gamma) \Rightarrow \gamma = \arctan\left(\frac{n_y}{n_x}\right)
\end{aligned}
$$

Be aware: We implicitly assumed $a_z \neq 0$ and $n_x \neq 0$. Matrices, that do not meet this assumption, need to be considered separately.

2. Conversion of Rotation Matrix to Quaternion

To determine the quaternion $\mathbf{q}$, the rotation axis and rotation angle need to be calculated.

Rotation axis: The rotation axis $\mathbf{x} \in \mathbb{R}^3$ fulfills $R_1\mathbf{x} = \mathbf{x}$. Thus, the rotation axis is the Eigen vector of R_1 corresponding to the Eigen value $\lambda_1 = 1$. Based on $(R_1 - \lambda_1 I)\mathbf{x} = (R_1 - I)\mathbf{x} = 0$, it follows:

$$
\begin{aligned}
-0.64\ x_1 + 0.48\ x_2 - 0.8\ x_3 &= 0 \\
-0.8\ x_1 - 0.4\ x_2 &= 0 \\
0.48\ x_1 + 0.64\ x_2 - 0.4\ x_3 &= 0
\end{aligned}
$$

Solving this equation system yields $\mathbf{x} = \pm\frac{1}{3}\begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$.

Rotation angle (Approach A): The rotation angle can be determined using the general formulation of a rotation matrix R for a rotation around a unit vector $\mathbf{v}$ by an angle α:

$$
R_{\mathbf{v},\alpha} = \begin{pmatrix} \cos(\alpha) + v_1^2(1-\cos(\alpha)) & v_1 v_2(1-\cos(\alpha)) - v_3\sin(\alpha) & v_1 v_3(1-\cos(\alpha)) + v_2\sin(\alpha) \\ v_2 v_1(1-\cos(\alpha)) + v_3\sin(\alpha) & \cos(\alpha) + v_2^2(1-\cos(\alpha)) & v_2 v_3(1-\cos(\alpha)) - v_1\sin(\alpha) \\ v_3 v_1(1-\cos(\alpha)) - v_2\sin(\alpha) & v_3 v_2(1-\cos(\alpha)) + v_1\sin(\alpha) & \cos(\alpha) + v_3^2(1-\cos(\alpha)) \end{pmatrix}
$$

With this, we get:

$$\text{Trace}(R) = 3\cos(\alpha) + \left(v_1^2 + v_2^2 + v_3^2\right)(1 - \cos(\alpha)) = 1 + 2\cos(\alpha)$$

In particular, for R_1 as defined in the exercise:

$$\text{Trace}(R_1) = 0.6 + 0.6 + 0.36 = 1.56 = 1 + 2\cos(\alpha) \Rightarrow \alpha = \pm 73.74^\circ$$

Rotation Angle (Approach B): We determine a vector $\mathbf{v}$ with $\mathbf{v} \cdot \mathbf{x} = 0$. This means, $\mathbf{v}$ is orthogonal to the rotation axis $\mathbf{x}$:

$$\mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{v}' = R_1 \cdot \mathbf{v} = \begin{pmatrix} 1.28 \\ 0.6 \\ 0.04 \end{pmatrix}$$

The rotation angle α is the angle between the vectors $\mathbf{v}$ and $\mathbf{v}'$:

$\cos(\alpha) = \frac{\mathbf{v} \cdot \mathbf{v}'}{\|\mathbf{v}\| \cdot \|\mathbf{v}'\|} = \frac{0.56}{\sqrt{2}^2} = 0.28 \Rightarrow \alpha = \arccos(0.28) \Rightarrow \alpha = \pm 1.287\,\text{rad} = \pm 73.74^\circ$

Quaternion: The quaternion $\mathbf{q}$ can be calculated from the rotation axis and rotation angle:

$$\mathbf{q} = \left(\cos\left(\frac{\alpha}{2}\right), \mathbf{x} \cdot \sin\left(\frac{\alpha}{2}\right)\right) = (0.8, 0.2, -0.4, -0.4)$$

Solution 2 (Homogene Matrizen)

1. T describes a rotation by 90° around the y-axis, and a translation by a vector $(5, 0, 0)^\top$.
2. The result is $\mathbf{v}' = (8, 2, -1)^\top$. Calculation:

$$T \cdot \mathbf{v} \quad = \begin{pmatrix} 0 & 0 & 1 & 5 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \\ -1 \\ 1 \end{pmatrix}$$

3. The inverse transformation matrix is calculated as follows:

$$T^{-1} \quad = \begin{pmatrix} n_x & n_y & n_z & -n^\top \mathbf{v} \\ o_x & o_y & o_z & -o^\top \mathbf{v} \\ a_x & a_y & a_z & -a^\top \mathbf{v} \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -5 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Solution 3 (Concatenation of Coordinate Transformations)

The local transformation matrices are

$$
\begin{aligned}
{}^{\text{init}}T_1 &= \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
{}^{1}T_2 &= \begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
{}^{2}T_3 &= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 2 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned}
$$

From this, the pose P_3 in BCS is calculated as follows:

$$
\begin{aligned}
P_3 &= T_{\text{init}} \cdot {}^{\text{init}}T_1 \cdot {}^{1}T_2 \cdot {}^{2}T_3 \\
&= \begin{pmatrix} 0 & -1 & 0 & 5 \\ 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot {}^{1}T_2 \cdot {}^{2}T_3 \\
&= \begin{pmatrix} 0 & -1 & 0 & 5 \\ 1 & 0 & 0 & 7 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot {}^{2}T_3 \\
&= \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 8 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 9 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned}
$$

Solution 4 (Distance Between Poses)

Translational distance Δt:

$$\begin{aligned}\Delta t = \|\mathbf{t}_{\text{Goal}} - \mathbf{t}_{\text{TCP}}\| &= \left\| \begin{pmatrix} 7 \\ 6 \\ 5 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\| = \left\| \begin{pmatrix} 6 \\ 4 \\ 2 \end{pmatrix} \right\| \\ &= \sqrt{6^2 + 4^2 + 2^2} = \sqrt{56} = 2\sqrt{14} = 7.48\end{aligned}$$

Rotational distance $\Delta\alpha$:

$$\begin{aligned}R_{\text{TCP,Goal}} &= R_{\text{TCP}}^{-1}\ R_{\text{Goal}} = R_{\text{TCP}}^{\top}\ R_{\text{Goal}} \\ &= \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{\top} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\end{aligned}$$

From exercise 1:

$$\text{Trace}(R) = 1 + 2\cos(\alpha)$$

Thus, for $R_{TCP,Goal}$:

$$\text{Trace}(R_{\text{TCP,Goal}}) = 0 = 1 + 2\cos(\Delta\alpha) \Rightarrow \Delta\alpha = 120°$$

Solution 5 (Quaternions)

1. $\mathbf{v} = 0 + 5i + 1j + 7k$
2. $\mathbf{q} = \left(\cos\frac{\phi}{2}, \mathbf{a}\sin\frac{\phi}{2}\right) = \cos\frac{\phi}{2} + k\cdot\sin\frac{\phi}{2} = \frac{1}{\sqrt{2}} + k\cdot\frac{1}{\sqrt{2}}$
   $\mathbf{q}^* = \frac{1}{\sqrt{2}} - k\cdot\frac{1}{\sqrt{2}}$
3. $$\begin{aligned}\mathbf{v}' = \mathbf{q}\mathbf{v}\mathbf{q}^* &= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}k\right)\cdot(5i + 1j + 7k)\cdot\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}k\right) \\ &= \left(\frac{5}{\sqrt{2}}i + \frac{1}{\sqrt{2}}j + \frac{7}{\sqrt{2}}k + \frac{5}{\sqrt{2}}ki + \frac{1}{\sqrt{2}}kj + \frac{7}{\sqrt{2}}k^2\right)\cdot\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}k\right) \\ &= \left(\frac{4}{\sqrt{2}}i + \frac{6}{\sqrt{2}}j + \frac{7}{\sqrt{2}}k - \frac{7}{\sqrt{2}}\right)\cdot\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}k\right) \\ &= 2i - 2ik + 3j - 3jk + \frac{7}{2}k - \frac{7}{2}k^2 - \frac{7}{2} + \frac{7}{2}k \\ &= -i + 5j + 7k\end{aligned}$$

   Result: $\mathbf{p}' = (-1, 5, 7)^{\top}$

4. At first, we calculate the angle between $\mathbf{q}_1$ and $\mathbf{q}_2$:

$$\cos\theta = \langle \mathbf{q}_1 \mid \mathbf{q}_2 \rangle = 0 \Rightarrow \theta = \frac{\pi}{2}$$

SLERP interpolation is calculated as:

$$\begin{aligned} \text{SLERP}(\mathbf{q}_1, \mathbf{q}_2, t) &= \frac{\sin((1-t)\frac{\pi}{2})}{\sin\frac{\pi}{2}} \cdot \mathbf{q}_1 + \frac{\sin(t\frac{\pi}{2})}{\sin\frac{\pi}{2}} \cdot \mathbf{q}_2 \\ &= \sin\left((1-t)\frac{\pi}{2}\right) \cdot \mathbf{q}_1 + \sin\left(t\frac{\pi}{2}\right) \cdot \mathbf{q}_2 \end{aligned}$$

For $t = \frac{1}{2}$:

$$\text{SLERP}\left(\mathbf{q}_1, \mathbf{q}_2, \frac{1}{2}\right) = \sin\frac{\pi}{4} \cdot \mathbf{q}_1 + \sin\frac{\pi}{4} \cdot \mathbf{q}_2 = \frac{1}{\sqrt{2}} \cdot \mathbf{q}_1 + \frac{1}{\sqrt{2}} \cdot \mathbf{q}_2 = \frac{1}{\sqrt{2}}i + \frac{1}{\sqrt{2}}j$$

Solution 6 (Quaternions)

Associativity and existance of an identity element follow from $\mathbb{H}$. Trivially, $\mathbb{S}^3 \subset \mathbb{H}$.

It remains to show that $\mathbb{S}^3$ is closed with respect to multiplication, and that an inverse element exists for every unit quaternion.

1. $\mathbf{q}_1, \mathbf{q}_2 \in \mathbb{S}^3 \Rightarrow \mathbf{q}_1 \cdot \mathbf{q}_2 \in \mathbb{S}^3$
2. $\mathbf{q} \in \mathbb{S}^3 \Rightarrow \mathbf{q}^{-1} \in \mathbb{S}^3$

Proof:

1. $\|\mathbf{q}_1 \cdot \mathbf{q}_2\|^2 = (\mathbf{q}_1\mathbf{q}_2) \cdot (\mathbf{q}_1\mathbf{q}_2)^* = \mathbf{q}_1(\mathbf{q}_2\mathbf{q}_2^*)\mathbf{q}_1^* = \mathbf{q}_1\|\mathbf{q}_2\|^2\mathbf{q}_1 = \mathbf{q}_1\mathbf{q}_1^* \cdot \|\mathbf{q}_2\|^2 = \|\mathbf{q}_1\|^2 \cdot \|\mathbf{q}_2\|^2 = 1$
2. $\|\mathbf{q}^{-1}\|^2 = \left\| \frac{\mathbf{q}^*}{\|\mathbf{q}\|^2} \right\|^2 = \|\mathbf{q}^*\|^2 = 1$

Solution 7 (Rotations and Machine Learning)

See exercise.